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On the topological interpretation of gravitational anomalies

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Abstract

We consider the mixed gravitational Yang–Mills anomaly as the coupling between the K-theory and K-homology of a C^*-algebra crossed product. The index theorem of Connes–Moscovici allows to compute the Chern character of the K-cycle by local formulae involving connections and curvatures. It gives a topological interpretation to the anomaly, in the sense of noncommutative algebras. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In a previous paper [9], we proved a formula computing the topological anomaly of gauge theories, in the very general framework on noncommutative geometry [4]. This formula reduces just to the pairing between the K-theory classes of loops in the gauge group, and some K-homology classes arising from abstract Dirac-type operators. This simple remark allows one to compare the usual BRS machinery with cyclic cohomology [4]. Both are nontrivial as *local* cohomologies, but we feel that cyclic cohomology is more suitable since it can be directly related to the *analytic* content of the Dirac-type operator via the Chern character, whereas BRS cohomology has no obvious link with index theory in general.

In this paper, we want to apply the topological anomaly formula in the mixed gravitational Yang–Mills case, i.e. when the gauge group is the crossed product of Yang–Mills transformations on a manifold X with a group of diffeomorphisms acting on X. Here the

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Chern character of the K-cycle involved takes its values in the cyclic cohomology of an algebra crossed product. The local index theorem of Connes and Moscovici [6] then expresses this Chern character in terms of Gelfand–Fuchs cohomology. We shall compute it by connections and curvatures, and see that it gives expressions very similar to the usual ones encountered in the (BRS) study of gravitational anomalies. However, there is an essential difference here: ordinary BRS methods deal with *Lie algebra* cohomology, whereas the characteristic classes for crossed products involve *group* cohomology. Of course both are related by van Est-type theorems, but we insist on the fact that group considerations can describe gravitational anomalies topologically, as the pairing of nontrivial cyclic cocycles with the K-theory of a (noncommutative) algebra crossed product.

This paper is organized as follows. In Section 2, we recall the anomaly formula in the case of Yang–Mills theories, with particular emphasis on its link with Bott periodicity, and we improve it by taking the diffeomorphisms into account.

In Section 3, we present a relatively self-contained collection of some classical results of Bott [2] and Haefliger [8] concerning equivariant cohomology and Gelfand–Fuchs cohomology, and apply it to the Connes–Moscovici index theorem for crossed products.

In Section 4, we illustrate these tools by the study of conformal transformations on a Riemann surface. This gives rise to a nontrivial cyclic cocycle, corresponding to a conformal anomaly.

2. The anomaly formula

2.1. Yang–Mills anomalies

Let X be an even-dimensional oriented smooth manifold, $C_0(X)$ the C^*-algebra of continuous complex-valued functions vanishing at infinity. We consider a K-cycle over X. For concreteness, let us take the signature complex: endow X with a Riemannian metric and let $H = H_+ \oplus H_-$ be the Hilbert space of L^2 differential forms on X, with $\mathbb{Z}_2$-graduation given by self- and anti-self-duality. The elliptic signature operator $D = d + d^*$ acts on a dense domain of H as an odd unbounded self-adjoint operator. The pair (H, D) defines in this way a K-homology class $[D] \in K^*(C_0(X))$.

A typical situation in quantum field theory is the following. Let N be a positive integer, and consider the group $G = U_N(C_c^\infty(X))$ of $N \times N$ unitary matrices with entries in the algebra of smooth compactly supported functions $C_c^\infty(X)$. It is the group of Yang–Mills transformations, with structure group U_N (if X is not compact, one should add a unit). G acts on the tensor product $H \otimes \mathbb{C}^N$ by even bounded endomorphisms. In general the elliptic operator comes equipped with an Yang–Mills connection A,

$$D_A = D + A, \tag{1}$$

which transforms under the gauge group according to the adjoint representation:

$$D_A \to u^{-1} D_A u, \quad u \in G, \qquad A \to u^{-1} A u + u^{-1}[D, u]. \tag{2}$$

As we work with $\mathbb{Z}_2$-graded spaces we adopt the usual matricial notation

$$D_A = \begin{pmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{pmatrix}, \qquad u = \begin{pmatrix} u_+ & 0 \\ 0 & u_- \end{pmatrix}, \qquad H = \begin{pmatrix} H_+ \\ H_- \end{pmatrix}. \tag{3}$$

Consider now the chiral action

$$S(\psi_+, \psi_-, A) = \langle \psi_-, D_A^+ \psi_+ \rangle, \quad \psi_\pm \in H_\pm. \tag{4}$$

If we quantize the fields $\psi_\pm$ according to the Fermi statistics, then the vacuum functional

$$Z(A) = \int [\mathrm{d}\psi]\, \mathrm{e}^{-S(\psi_+, \psi_-, A)} \tag{5}$$

is simply given by a regularized determinant $\det D_A^+$, see [10] (for the Bose statistics one takes the inverse determinant). In general it is not invariant under the gauge group, i.e.

$$\det D_A^+ \neq \det(u^{-1} D_A u)^+. \tag{6}$$

Define the loop group G^{S^1} as the set of smooth maps

$$g : S^1 \to U_N(C_c^\infty(X)) \tag{7}$$

with base-point as the identity. The product is pointwise. Let $t \in [0, 2\pi)$ be the coordinate on S^1. Given a loop $g \in G^{S^1}$ the determinant $Z(t) = \det(g^{-1}(t) D_A g(t))^+$ is an invertible $\mathbb{C}$-valued function on S^1 and the topological anomaly is just the winding number

$$w = \frac{1}{2\pi \mathrm{i}} \int_{S^1} \frac{\mathrm{d}Z(t)}{Z(t)} \in \mathbb{Z}. \tag{8}$$

The anomaly formula of Perrot [9] relates it to the K-cycle $[D]$ as follows. First the loop group G^{S^1} may be identified with $U_N(C_c^\infty(S^1 \times X))$. Then any element g in the loop group determines a class $[g]$ of the K-theory group $K_1(C_0(S^1 \times X))$. The operator

$$Q = \begin{pmatrix} \mathrm{i}\partial_t & D_A^- \\ D_A^+ & -\mathrm{i}\partial_t \end{pmatrix}, \tag{9}$$

acting on sections of the Hilbert bundle $H \times S^1$, represents an odd K-cycle for the C^*-algebra $C_0(S^1 \times X)$. Let $\mathrm{ch}_*(Q)$ be its Chern character [4] in the odd cyclic cohomology of $C_c^\infty(S^1 \times X)$. One has a well-defined, homotopy-invariant, integral pairing

$$w = \langle [g], \mathrm{ch}_*(Q) \rangle \in \mathbb{Z} \tag{10}$$

computing the value of the topological (Yang–Mills) anomaly on the loop g [9].

Observe that in (10) any reference to the connection A disappears. It is a purely topological formula involving the K-homology class of Q and the K-theory element $[g]$. In particular, if the first homotopy group of $U_N(C_0(X))$ is zero, then any loop g is contractible and its image $[g]$ vanishes in $K_1(C_0(X \times S^1))$. By increasing N, we may eventually choose nontrivial loops detected by Q, which is really the essence of K-theory.

In [9], we established the anomaly formula in a more general setting, allowing for X to be a noncommutative "space" described by an associative algebra $\mathcal{A}$ together with an even Fredholm module (H, D) [4] playing the role of the previous elliptic operator. Our goal in the following is to apply these ideas in the case of gravitational theories, where the gauge group contains the diffeomorphisms $\mathrm{Diff}(X)$ of the manifold X. The relevant space is the groupoid $X \rtimes \mathrm{Diff}(X)$, which is highly noncommutative in nature.

2.2. The gravitational case

We thus implement the above construction by taking the diffeomorphisms of X into account. The group of mixed Yang–Mills $\rtimes$ gravitational transformations is the crossed product $U_N(C_c^\infty(X)) \rtimes \mathrm{Diff}(X)$, which lies in the matrix algebra of $\mathcal{A} = C_c^\infty(X) \rtimes \mathrm{Diff}(X)$. The associative algebra $\mathcal{A}$ is generated by the symbols

$$a = fU_\psi^*, \quad f \in C_c^\infty(X), \quad \psi \in \mathrm{Diff}(X) \tag{11}$$

with product rule

$$(f_1 U_{\psi_1}^*)(f_2 U_{\psi_2}^*) = f_1(f_2 \circ \psi_1) U_{\psi_2\psi_1}^*, \tag{12}$$

where $f_2 \circ \psi_1$ is the pullback of f_2 by ψ_1. Since we are mostly concerned with K-theory, we shall enlarge this group to all invertible elements $\mathrm{Gl}_N(\mathcal{A})$ as well. Thus we are dealing with transformations involving matrices of diffeomorphisms. The physical interpretation seems obscure at first sight, but our motivation comes from the fact that the group $\mathrm{Diff}(X)$ does not generally contain enough nontrivial loops. We shall see in the following that provided we consider matrix algebras, in the same philosophy as in the Yang–Mills case, the anomaly formula detects nontrivial topological objects related to diffeomorphisms. Let us explain carefully the construction in this case.

Let $\mathrm{Diff}(S^1, X)$ denote the subgroup of $\mathrm{Diff}(S^1 \times X)$ consisting in diffeomorphisms ψ such that

$$\mathrm{pr} \circ \psi = \mathrm{pr}, \tag{13}$$

where $\mathrm{pr} : S^1 \times X \to S^1$ is the projection onto the first factor. Then $\mathrm{Diff}(S^1, X)$ plays the role of the loop group of $\mathrm{Diff}(X)$. Thus we identify the loop group of $\mathrm{Gl}_N(\mathcal{A})$ with $\mathrm{Gl}_N(C_c^\infty(S^1 \times X) \rtimes \mathrm{Diff}(S^1, X))$.

From now on put $M = S^1 \times X$. For the sake of definiteness, let Γ be a discrete countable subgroup of $\mathrm{Diff}(S^1, X)$. We choose the loops as elements of $\mathrm{Gl}_N(C_c^\infty(M) \rtimes \Gamma)$ and consider their images in the group K_1 of the C^*-algebra $C_0(M) \rtimes \Gamma$. As before we would like to evaluate these K-theory classes on some K-cycle Q. The previous signature operator is not suitable in this case because it does not define a K-cycle for $C_0(M) \rtimes \Gamma$. This problem is solved as in [5] by passing to the bundle P over M, whose fiber at $x \in M$ is the set of all Euclidean metrics on the tangent space T_xM. Γ acts canonically on P and the K-theory of $C_0(M) \rtimes \Gamma$ lifts through the Thom map of Connes [3]

$$\beta : K_*(C_0(M) \rtimes \Gamma) \to K_*(C_0(P) \rtimes \Gamma). \tag{14}$$

On this bundle P of metrics one can construct a differential operator Q representing a K-cycle for $C_0(P) \rtimes \Gamma$, playing the role of the signature class [5]. If we let $\mathrm{ch}_*(Q)$ be its Chern character in the cyclic cohomology of $C_c^\infty(P) \rtimes \Gamma$, the anomaly formula amounts to the computation of

$$\langle \beta([g]), \mathrm{ch}_*(Q) \rangle, \quad [g] \in K_1(C_0(M) \rtimes \Gamma) \tag{15}$$

for any loop g. In the following sections, we shall use the index theorem of Connes and Moscovici [6] to express $\mathrm{ch}_*(Q)$ as an equivariant cohomology class. The latter is constructed from connections and curvatures in great analogy with the usual expressions of gravitational anomalies found in the literature. This together with the nontriviality of $\mathrm{ch}_*(Q)$ gives an interesting K-theoretical interpretation of these anomalies.

2.3. Remark on Bott periodicity

Note that in the context of C^*-algebras, the pure Yang–Mills anomaly has a simple interpretation in terms of Bott periodicity [1, Section 9]. Indeed the set of homotopy classes of loops in $U_\infty(C_0(X))$ with base-point 1 is isomorphic to the group K_1 of the suspension of $C_0(X)$. Moreover, the product in the loop group of $U_\infty(C_0(X))$ can eventually be taken as the concatenation of loops, so that

$$\pi_1(U_\infty(C_0(X))) \simeq K_1(C_0(X \times \mathbb{R})), \tag{16}$$

and Bott periodicity stands for the isomorphism

$$\theta : K_0(C_0(X)) \to K_1(C_0(X \times \mathbb{R})). \tag{17}$$

Also the Chern character of the differential operator (9) is just the cup product

$$\mathrm{ch}_*(Q) = \mathrm{ch}_*(D) \# [S^1], \tag{18}$$

between $\mathrm{ch}_*(D)$ in the cyclic cohomology of $C_c^\infty(X)$ and the fundamental class of the circle. Hence one has an equality (see, e.g. [4, p. 225, Proposition 3c])

$$\langle [g], \mathrm{ch}_*(Q) \rangle = \langle \theta^{-1}([g]), \mathrm{ch}_*(D) \rangle \tag{19}$$

for any loop $g \in G^{S^1}$. It follows that the evaluation of the Yang–Mills anomaly on a loop in the gauge group $U_\infty(C_c^\infty(X))$ is *equivalent* to the coupling between the K-theory of X and the K-homology class $[D]$. This interpretation does not hold true for the gravitational anomaly because $C_0(M) \rtimes \Gamma$ is not the suspension of a C^*-algebra in general.

3. Characteristic classes for crossed products

In this section, we recall basic facts about equivariant cohomology and Gelfand–Fuchs cohomology. Most of this material can be found in [2,8]. It allows to compute the characteristic classes of the crossed product $M \rtimes \Gamma$ appearing in the Connes–Moscovici index theorem [6], in terms of connections and curvatures.

3.1. Equivariant cohomology

Let M be an oriented manifold, and Γ a discrete group acting on M by orientation-preserving diffeomorphisms. In the following, we will not distinguish an element g of Γ and the corresponding diffeomorphism.

The space of homogeneous cochains of bidegree n, m is zero if $n < 0$ or $m < 0$ otherwise it is the space $C^{n,m}(M)$ of maps u from Γ^{n+1} to the differential forms $\Omega^m(M)$ of degree m on M, subject to the equivariance condition

$$u(g_0 g, \dots, g_n g) = u(g_0, \dots, g_n) \circ g, \quad g_i, g \in \Gamma, \tag{20}$$

where $\circ g$ denotes the pullback by the diffeomorphism g. On the complex $C^{*,*}$ one defines two differentials. The first one $\delta : C^{n,m} \to C^{n+1,m}$ is the simplicial differential

$$(\delta u)(g_0, \dots, g_{n+1}) = (-)^m \sum_{i=0}^{n+1} (-)^i u(g_0, \dots, \overset{\vee}{g_i}, \dots, g_{n+1}), \tag{21}$$

where $^\vee$ denotes omission. The second one $\mathrm{d} : C^{n,m} \to C^{n,m+1}$ is the de Rham coboundary

$$(\mathrm{d}u)(g_0, \dots, g_n) = \mathrm{d}(u(g_0, \dots, g_n)). \tag{22}$$

The signs are chosen so that $\mathrm{d}^2 = \delta^2 = \mathrm{d}\delta + \delta\mathrm{d} = 0$. Geometrically, the total complex $(C^{*,*}, \mathrm{d}+\delta)$ describes the complex of cochains on the homotopy quotient $M_\Gamma = M \times_\Gamma E\Gamma$. By definition its cohomology is the *equivariant cohomology* $H^*(M_\Gamma)$ of M.

It will be convenient for us to consider the following ring structure on homogeneous cochains. For $u \in C^{n,m}$ and $v \in C^{p,q}$, the product $uv \in C^{n+p,m+q}$ is

$$(uv)(g_0, \dots, g_{n+p}) = (-)^{nq} u(g_0, \dots, g_n) v(g_n, \dots, g_{n+p}). \tag{23}$$

This product is associative and compatible with equivariance. Moreover, the Leibniz rule is satisfied:

$$\mathrm{d}(uv) = \mathrm{d}uv + (-)^{n+m} u\, \mathrm{d}v, \qquad \delta(uv) = \delta uv + (-)^{n+m} u\, \delta v \tag{24}$$

with $n + m$ the total degree of u. Thus $(C^{n,m}, \mathrm{d} + \delta)$ is a graded differential algebra.

Recall also [8] that the above complex of homogeneous cochains is isomorphic to the complex of group cochains $C^*(\Gamma, \Omega^*(M))$ with coefficients in the differential forms of M. To $u \in C^{n,m}(M)$ corresponds the group cochain $f \in C^n(\Gamma, \Omega^m(M))$:

$$f(g_1, \dots, g_n) := u(g_1, \dots, g_n, g_2, \dots, g_n, \dots, g_n, 1), \tag{25}$$

and the associated coboundary operator $\delta : C^n(\Gamma, \Omega^m) \to C^{n+1}(\Gamma, \Omega^m)$ reads

$$\delta f(g_1, \dots, g_{n+1}) = f(g_2, \dots, g_{n+1}) + \sum_{i=1}^{n} (-)^i f(g_1, \dots, g_i, g_{i+1}, \dots, g_{n+1}) + (-)^{n+1} f(g_1, \dots, g_n) \circ g_{n+1}. \tag{26}$$

3.2. Jet bundles

Let n be the dimension of the manifold M. Let J_k^+ be the space of k-jets of orientation-preserving diffeomorphisms

$$j : \mathbb{R}^n \to M \tag{27}$$

from a neighborhood of 0 in $\mathbb{R}^n$ to M. Given a local coordinate system $\{x^\mu\}$, $\mu = 1, \dots, n$ on M, J_k^+ has coordinates $\{x^\mu, y_i^\mu, y_{i_1 i_2}^\mu, \dots, y_{i_1,\dots,i_k}^\mu\}$ corresponding to the jet

$$j^\mu(u) = x^\mu + y_i^\mu u^i + \frac{1}{2} y_{ij}^\mu u^i u^j + \cdots + \frac{1}{k!} y_{i_1,\dots,i_k}^\mu u^{i_1}, \dots, u^{i_k}, \quad u \in \mathbb{R}^n. \tag{28}$$

In particular, the matrix (y_i^μ) belongs to $\mathrm{Gl}_n^+(\mathbb{R})$, and the real numbers $y_{i_1,\dots,i_l}^\mu$ are symmetric in low indices.

J_k^+ is a principal bundle over M with structure group G^k consisting in the set of k-jets h fixing 0: $h^\mu(0) = 0$. The right action of G^k on J_k^+ is simply the composition of jets:

$$j \to j \circ h, \quad j \in J_k^+, \quad h \in G^k. \tag{29}$$

Since any $(k+1)$-jet yields a k-jet, J_{k+1}^+ is a principal bundle over J_k^+ with structure group the kernel of the projection $G^{k+1} \to G^k$. We get in this way a tower of bundles

$$\cdots \to J_k^+ \to \cdots \to J_1^+ \to M. \tag{30}$$

We write the inverse limit J_∞^+. Note that J_1^+ is the bundle of oriented frames on M.

The action of Γ on M lifts on J_k^+ by left composition of jets

$$j \to g \circ j, \quad j \in J_k^+, \quad g \in \Gamma, \tag{31}$$

and clearly commutes with G^k. In particular, the group $\mathrm{SO}_n \subset \mathrm{Gl}_n^+$ sits in G^k as a maximal compact subgroup and Γ still acts on the quotient $P_k = J_k^+/\mathrm{SO}_n$, which is a bundle with contractible fiber over M. The action of an element $a \in \mathrm{SO}_n$ is given by the right composition by the jet

$$h^i(u) = a_j^i u^j, \quad u \in \mathbb{R}^n, \tag{32}$$

where $a = (a_j^i)$ is a matrix in SO_n. Explicitly the vertical coordinates of a point in J_k^+ change according to the rule

$$y_{i_1,\dots,i_l}^\mu \to y_{j_1,\dots,j_l}^\mu a_{i_1}^{j_1}, \dots, a_{i_l}^{j_l}. \tag{33}$$

Thus one has a tower of bundles with contractible fiber

$$\cdots \to P_k \to \cdots \to P_1 \to M \tag{34}$$

with inverse limit P_∞. Remark that P_1 is the bundle of metrics over M. Since Γ lifts on each P_k, the homotopy quotient $P_{k,\Gamma} = P_k \times_\Gamma E\Gamma$ is a bundle over M_Γ with contractible fiber, which induces an isomorphism in equivariant cohomology

$$H^*(P_{k,\Gamma}) \simeq H^*(M_\Gamma), \tag{35}$$

and also for the limit $H^*(P_{\infty,\Gamma})$.

3.3. Gelfand–Fuchs cohomology

Given a coordinate chart $\{x^\mu, y_i^\mu, \dots, y^\mu_{i_1,\dots,i_k}, \dots\}$, we identify *locally* J^+_∞ with the pseudogroup of all diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$. Its Lie algebra $\mathfrak{a}$ corresponds to the formal vector fields of $\mathbb{R}^n$. Let $\Omega^*_{\text{inv}}(J^+_\infty)$ be the complex of invariant forms under the left action of $\mathrm{Diff}(M)$ on jets by composition

$$j \to \varphi \circ j, \quad \varphi \in \mathrm{Diff}(M). \tag{36}$$

It is naturally isomorphic to the complex of Lie algebra cochains $C^*(\mathfrak{a}, \mathbb{R})$. An algebraic basis of $\mathrm{Diff}(M)$-invariant forms on J^+_∞ is provided by expanding the Maurer–Cartan form "$j^{-1} \circ \mathrm{d}j$" in powers of $u \in \mathbb{R}^n$:

$$(j^{-1} \circ \mathrm{d}j)^i(u) = \theta^i + \theta^i_j u^i + \frac{1}{2}\theta^i_{jk} u^j u^k + \cdots + \frac{1}{k!}\theta^i_{j_1,\dots,j_k} u^{j_1}, \dots, u^{j_k} + \cdots . \tag{37}$$

Due to the $\mathrm{Diff}(M)$-invariance, the one-forms θ are globally defined on J^+_∞. Actually, $\theta^i_{j_1,\dots,j_k}$ is already defined on J^+_{k+1}. For example

$$\theta^i = (y^{-1})^i_\mu \, \mathrm{d}x^\mu \tag{38}$$

lies on J^+_1 (here $((y^{-1})^i_\mu)$ is the inverse matrix of (y^μ_i)),

$$\theta^i_j = (y^{-1})^i_\mu \, \mathrm{d}y^\mu_j - y^\mu_{jk} (y^{-1})^i_\mu (y^{-1})^k_\nu \, \mathrm{d}x^\nu \tag{39}$$

lies on J^+_2, and so on. Thus the Gelfand–Fuchs cohomology $H^*(\mathfrak{a}, \mathbb{R})$ is naturally isomorphic to the cohomology of invariant forms $H^*(\Omega^*_{\text{inv}}(J^+_\infty))$. It is computed as follows [7]. The group $\mathrm{Gl}_n(\mathbb{R})$ acts on $\mathbb{R}^n$ be linear diffeomorphisms. Let $\mathfrak{g} \subset \mathfrak{a}$ be its the Lie algebra. The Weil algebra associated to $\mathfrak{g}$ is the tensor product

$$W = \wedge \mathfrak{g}^* \otimes S(\mathfrak{g}^*) \tag{40}$$

of the exterior algebra on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$, by the symmetric algebra $S(\mathfrak{g}^*)$. W is a graded differential algebra: it is generated by the elements of degree 1

$$\omega^i_j \in \wedge^1 \mathfrak{g}^* \tag{41}$$

of the canonical basis of $\mathfrak{g}^*$ and

$$\Omega^i_j \in S^1(\mathfrak{g}^*) \tag{42}$$

of degree 2. A differential d_W is uniquely defined by

$$\mathrm{d}_W \omega^i_j = \Omega^i_j - \omega^i_k \omega^k_j, \qquad \mathrm{d}_W \Omega^i_j = \Omega^i_k \omega^k_j - \omega^i_k \Omega^k_j. \tag{43}$$

Next we consider θ^i_j as the coefficients of a connection 1-form on $\mathfrak{a}$ with values in $\mathfrak{g}$, and its curvature

$$R^i_j = \mathrm{d}\theta^i_j + \theta^i_k \theta^k_j. \tag{44}$$

Then from Chern–Weil theory one has a morphism

$$\psi : W \to \Omega^*_{\mathrm{inv}}(J^+_\infty), \tag{45}$$

which sends ω^i_j onto θ^i_j and Ω^i_j onto R^i_j. Furthermore, the 2-form R^i_j is proportional to $\mathrm{d}x^\mu$ and hence any polynomial in R of degree $>n$ vanishes. It follows that ψ factorizes through W_n, the quotient of W by the differential ideal generated by the elements in $S(\mathfrak{g}^*)$ of degree $>2n$. The first result of Gelfand–Fuchs the following theorem [7].

Theorem 1. *The map* $\psi : W_n \to \Omega^*_{\mathrm{inv}}(J^+_\infty)$ *induces an isomorphism in cohomology.*

This theorem also admits a version relative to the action of SO_n on $\mathfrak{a}$. The complex of SO_n-basic cochains $C^*(\mathfrak{a}, \mathrm{SO}_n)$ is naturally isomorphic to the invariant forms on $P_\infty = J^+_\infty/\mathrm{SO}_n$. Since $\mathrm{SO}_n \subset \mathrm{Gl}_n$, W_n is an SO_n-algebra. Let $W\,\mathrm{SO}_n$ be its subalgebra of basic elements relative to the action of SO_n. Then ψ maps $W\,\mathrm{SO}_n$ to $\Omega^*_{\mathrm{inv}}(P_\infty)$ and one has the following theorem [7].

Theorem 2. *The map* ψ *induces an isomorphism*

$$H^*(W\,\mathrm{SO}_n) \simeq H^*(\Omega^*_{\mathrm{inv}}(P_\infty)). \tag{46}$$

Next we want to send these classes into the equivariant cohomology of M. Remark that there is an injection

$$i : \Omega^m(P_\infty) \to C^{0,m}(P_\infty), \tag{47}$$

which to any (not necessarily invariant) differential form α on P_∞ associates the homogeneous 0-cochain

$$\alpha(g_0) := \alpha \circ g_0 \quad \forall g_0 \in \Gamma. \tag{48}$$

It is clear that under this map the image of a closed form in $\Omega^*_{\mathrm{inv}}(P_\infty)$ is both d- and δ-closed, and hence defines an equivariant cohomology class on $P_{\infty,\Gamma}$. Thus one gets a canonical map

$$H^*(W\,\mathrm{SO}_n) \to H^*(P_{\infty,\Gamma}) \simeq H^*(M_\Gamma). \tag{49}$$

Note finally that the image of $W\,\mathrm{SO}_n$ by ψ lives in $P_2 = J^+_2/\mathrm{SO}_n$ since the forms θ^i_j and $R^i_j = \mathrm{d}\theta^i_j + \theta^i_k\theta^k_j$ are defined on J^+_2. It is then sufficient to work on P_2 instead of P_∞.

3.4. Computation of $H^*(W\,\mathrm{SO}_n)$

We restrict to the case of a manifold M of odd dimension n. In the truncated Weil algebra W_n, the Chern classes c_i, $i = 1, \dots, n$, correspond to the terms of degree $2i$ in the determinant of the $n \times n$ matrix $1 + \Omega$. In particular,

$$c_1 = \Omega^i_i, \qquad c_2 = \tfrac{1}{2}((\Omega^i_i)^2 - \Omega^i_j\Omega^j_i), \qquad c_n = \det\Omega. \tag{50}$$

For i odd one can choose an element u_i of degree $2i-1$ in $W\,\mathrm{SO}_n$ such that $\mathrm{d}_W u_i = c_i$. Let $E(u_1, u_3, \ldots, u_n)$ be the exterior algebra in the u_i, i odd $\leq n$, and $\mathbb{R}[c_1, c_2, \ldots, c_n]$ the algebra of polynomials in all the c_i quotiented by the ideal of elements of degree strictly higher than $2n$. The tensor product

$$WU_n = \mathbb{R}[c_1, \ldots, c_n] \otimes E(u_1, \ldots, u_n) \quad (51)$$

is endowed with the differential d such that $\mathrm{d}u_i = c_i$. Then one has the following theorem [7].

Theorem 3. *The inclusion $WU_n \to W\,\mathrm{SO}_n$ induces an isomorphism in cohomology.*

In particular, if we define the Pontrjagin classes

$$p_i = c_{2i} \quad \forall i \leq \tfrac{1}{2}(n-1), \quad n \text{ odd}, \quad (52)$$

then $H^*(W\,\mathrm{SO}_n)$ always contains the polynomial algebra $\mathbb{R}[p_1, p_2, \ldots]_{\mathrm{trunc}}$ in the p_i's truncated by the elements of degree $>2n$.

3.5. The Connes–Moscovici index theorem

Let $P = P_1$ be the bundle of metrics over the odd-dimensional manifold M. On P the hypoelliptic signature operator Q of Connes and Moscovici [5] defines a K-cycle for the algebra $C_0(P) \rtimes \Gamma$. By Connes [4, Section 3.2.δ] one has an injective map

$$\Phi : H^*(P \times_\Gamma E\Gamma) \hookrightarrow HC^*_{\mathrm{per}}(C^\infty_c(P) \rtimes \Gamma) \quad (53)$$

from equivariant cohomology to the periodic cyclic cohomology of the crossed product $C^\infty_c(P) \rtimes \Gamma$. The index theorem of Connes and Moscovici [6] states that the Chern character $\mathrm{ch}_*(Q) \in HC^*_{\mathrm{per}}(C^\infty_c(P) \rtimes \Gamma)$ is in the range of Gelfand–Fuchs cohomology. Actually $\mathrm{ch}_*(Q)$ has a preimage in the Pontrjagin ring $\mathbb{R}[p_1, p_2, \ldots]_{\mathrm{trunc}}$.

If we apply this construction to the situation of Section 2, where $M = S^1 \times X$ and Γ is a loop group of diffeomorphisms on X, the complete computation of the anomaly formula (15)

$$\langle \beta([g]), \mathrm{ch}_*(Q) \rangle, \qquad [g] \in K_1(C_0(M) \rtimes \Gamma) \quad (54)$$

yields an expression containing the image of the Pontrjagin classes in $H^*(M_\Gamma)$ and other characteristic classes accounting for the Thom isomorphism β. In the following section, we compute the image of the Pontrjagin ring in the particular case of Riemann surfaces and conformal transformations, and see that the result looks like familiar gravitational anomalies. The same holds clearly true in the general case.

4. Application to Riemann surfaces

Let us have a look at the simplest example. We take M as the product of S^1 by a Riemann surface Σ. We view it as a trivial fiber bundle over S^1 with fiber Σ. Let Γ be a discrete (pseudo)group of orientation preserving diffeomorphisms on M fulfilling the two conditions:

1. Each fiber Σ over S^1 is globally Γ-invariant.
2. The restriction of Γ to a fiber is a conformal transformation of Σ.

Thus according to Section 2 an element of Γ is a loop of conformal transformations of Σ. Choose a local coordinate system $(z, \bar{z})$ related to the complex structure of Σ, and let $t \in [0, 2\pi)$ be the variable on S^1. For any $g \in \Gamma$, we write

$$z \circ g = Z_g, \qquad \bar{z} \circ g = \bar{Z}_g, \qquad t \circ g = t. \tag{55}$$

The jet bundles J_k^+ have local coordinates $(x^\mu, y_i^\mu, \dots, y^\mu_{i_1,\dots,i_k})$, where the indices $\mu, i_l, \dots,$ can assume any of the three values $(z, \bar{z}, t)$. Of course x^μ are identified with the coordinates on M:

$$x^z = z, \qquad x^{\bar{z}} = \bar{z}, \qquad x^t = t. \tag{56}$$

Since the (real) dimension of M is $n = 3$ the Pontrjagin ring of the Gelfand–Fuchs cohomology $H^*(W\,\mathrm{SO}_3)$ only contains the unit 1 and the first Pontrjagin class $p_1 = c_2$. From the last section, we know that p_1 is represented by a closed Γ-invariant 4-form on the bundle $P_2 = J_2^+/\mathrm{SO}_3$, explicitly given in terms of the tautological curvature R^i_j, $i, j = (z, \bar{z}, t)$:

$$p_1 = \tfrac{1}{2}((R^i_i)^2 - R^i_j R^j_i) \in \Omega^4_{\mathrm{inv}}(P_2). \tag{57}$$

We denote $\hat{p}_1$ its image in $H^4(P_{2,\Gamma}) \simeq H^4(M_\Gamma)$.

4.1. Restriction to a subbundle

Since Γ is a group of conformal transformations of the fibers Σ leaving t invariant, one can restrict the geometry to the subbundle $\tilde{J}_2^+$ of J_2^+ consisting in holomorphic 2-jets

$$u \in \mathbb{R}^3 \to j(u) \in M, \tag{58}$$

which read in coordinates

$$\begin{aligned} j^z(u) &= z + y^z_z u^z + y^z_t u^t + \tfrac{1}{2} y^z_{zz} u^z u^z + y^z_{zt} u^z u^t + \tfrac{1}{2} y^z_{tt} u^t u^t, \\ j^{\bar{z}}(u) &= \bar{z} + y^{\bar{z}}_{\bar{z}} u^{\bar{z}} + y^{\bar{z}}_t u^t + \tfrac{1}{2} y^{\bar{z}}_{\bar{z}\bar{z}} u^{\bar{z}} u^{\bar{z}} + y^{\bar{z}}_{\bar{z}t} u^{\bar{z}} u^t + \tfrac{1}{2} y^{\bar{z}}_{tt} u^t u^t, \\ j^t(u) &= t + u^t. \end{aligned} \tag{59}$$

Then the 2-jets of the elements of Γ are contained in $\tilde{J}_2^+$. $\tilde{J}_2^+$ is a principal bundle over M, whose structure group contains SO_2 as a maximal compact subgroup. The action of SO_2 is obtained by the right composition

$$j \in \tilde{J}_2^+ \to j \circ h \in \tilde{J}_2^+, \tag{60}$$

where h is the jet of the rotation by an angle α:

$$h^z(u) = \mathrm{e}^{\mathrm{i}\alpha} u^z, \qquad h^{\bar{z}}(u) = \mathrm{e}^{-\mathrm{i}\alpha} u^{\bar{z}}, \qquad h^t(u) = u^t. \tag{61}$$

Thus $\tilde{P}_2 = \tilde{J}_2^+/\mathrm{SO}_2$ is a Γ-bundle over M with contractible fiber so that $H^*(\tilde{P}_{2,\Gamma}) = H^*(M_\Gamma)$. Moreover, $\tilde{P}_2$ is a Γ-invariant subbundle of P_2 and the injection $\tilde{P}_2 \to P_2$ is a homotopy equivalence. Now $\hat{p}_1 \in H^*(M_\Gamma)$ may equivalently be represented by a closed invariant form on $\tilde{P}_2$ corresponding to the pullback of (57). One computes that the pullbacks of the curvature coefficients R^i_j are nonzero only for $R^z_z, R^z_t, R^{\bar{z}}_{\bar{z}}, R^{\bar{z}}_t$, hence

$$\hat{p}_1 = \tfrac{1}{2}((R^z_z + R^{\bar{z}}_{\bar{z}})^2 - (R^z_z)^2 - (R^{\bar{z}}_{\bar{z}})^2) = R^z_z R^{\bar{z}}_{\bar{z}} \tag{62}$$

is the pullback of $\hat{p}_1$ on $\tilde{J}_2^+$, and is SO_2-basic. In terms of the tautological connection θ^i_j (Eq. (25)) on $\tilde{J}_2^+$ one has $R^z_z = \mathrm{d}\theta^z_z$ with

$$\theta^z_z = (y^{-1})^z_z\,\mathrm{d}y^z_z - y^z_{zz}(y^{-1})^z_z((y^{-1})^z_z\,\mathrm{d}z + (y^{-1})^z_t\,\mathrm{d}t) - y^z_{zt}(y^{-1})^z_z\,\mathrm{d}t, \tag{63}$$

and similarly for $R^{\bar{z}}_{\bar{z}}$. In the following, we shall write R (resp. $\bar{R}$) instead of R^z_z (resp. $R^{\bar{z}}_{\bar{z}}$) and θ (resp. $\bar{\theta}$) instead of θ^z_z (resp. $\theta^{\bar{z}}_{\bar{z}}$). Remark that the 1-form $\theta + \bar{\theta}$ is SO_2-basic, which implies that the cohomology class of $R + \bar{R}$ in the Γ-invariant forms on $\tilde{P}_2$ is zero. Thus $R\bar{R}$ is cohomologous to $-R^2$ and we shall keep the latter as a representative of $\hat{p}_1$.

It is possible now to express $\hat{p}_1$ as an equivariant cocycle on M_Γ. Choose a Kähler metric $\rho(z,\bar{z})\,\mathrm{d}z \otimes \mathrm{d}\bar{z}$ on Σ. Then the associated connection on $\tilde{J}_2^+$ is the globally defined (not Γ-invariant) 1-form

$$\omega = (y^{-1})^z_z\,\mathrm{d}y^z_z + \mathrm{d}z\,\partial_z \ln \rho. \tag{64}$$

Of course it corresponds to the ^z_z component of the connection form associated with ρ on the frame bundle. We shall regard it as an equivariant cochain on $\tilde{J}_2^+$ through the inclusion $\Omega^1(\tilde{J}_2^+) \to C^{0,1}(\tilde{J}_2^+)$. The equivariant curvature $\Omega = (\mathrm{d}+\delta)\omega$ is an element of $C^{0,2}(\tilde{J}_2^+) \oplus C^{1,1}(\tilde{J}_2^+)$:

$$\begin{aligned} \Omega(g_0, g_1) &= \delta\omega(g_0, g_1) = -\omega \circ g_1 + \omega \circ g_0, \\ \Omega(g_0) &= \mathrm{d}\omega(g_0) = \mathrm{d}\omega \circ g_0, \quad g_i \in \Gamma. \end{aligned} \tag{65}$$

In fact, Ω lives in M_Γ. Indeed, for $g_i \in \Gamma$, let Z'_i denote the function $\partial_z(z \circ g_i)$. One has (with $\partial = \mathrm{d}z\,\partial_z$)

$$\omega \circ g_i = (y^{-1})^z_z\,\mathrm{d}y^z_z + \mathrm{d}\ln Z'_i + (\partial \ln \rho) \circ g_i, \tag{66}$$

so that

$$\begin{aligned} \Omega(g_0, g_1) &= \mathrm{d}\ln Z'_0 - \mathrm{d}\ln Z'_1 + (\partial \ln \rho) \circ g_0 - (\partial \ln \rho) \circ g_1, \\ \Omega(g_0) &= -(\partial\bar{\partial} \ln \rho) \circ g_0. \end{aligned} \tag{67}$$

Here $\partial\bar{\partial} \ln \rho$ is the curvature 2-form of the Kähler metric. Using the multiplicative structure on equivariant cohomology (Section 3), we consider the cocycle $-\Omega^2$. It is cohomologous to $-R^2$ in $H^4(\tilde{P}_{2,\Gamma})$, indeed

$$\Omega^2 - R^2 = \tfrac{1}{2}(\mathrm{d}+\delta)((\omega-\theta)(\Omega+R)+(\Omega+R)(\omega-\theta)), \tag{68}$$

and $\omega-\theta$ is an SO_2-basic equivariant 1-form on $\tilde{J}_2^+$. Thus we have proved the following theorem.

Theorem 4. *The equivariant* 4*-cocycle* $-\Omega^2$ *represents the image of* $p_1 \in H^*(W\,SO_3)$ *in* $H^4(M_\Gamma)$.

4.2. Link with conformal anomalies

Using formula (25), we can express $\hat{p}_1$ as a group cocycle $\tilde{p}_1$ in $C^1(\Gamma, \Omega^3(M)) \oplus C^2(\Gamma, \Omega^2(M))$:

$$\tilde{p}_1(g) = \hat{p}_1(g,1), \qquad \tilde{p}_1(g_1,g_2) = \hat{p}_1(g_1g_2, g_2, 1). \tag{69}$$

The first component $\tilde{p}_1(g)$ is related to conformal anomalies as follows. Let $g : S^1 \to \mathrm{Diff}(\Sigma)$ be a loop of conformal transformations of Σ, i.e., $g \in \mathrm{Diff}(S^1, \Sigma)$ according to the notations of Section 2. Then $\tilde{p}_1(g)$ is a 3-form on $M = S^1 \times \Sigma$:

$$\begin{aligned}\tilde{p}_1(g) &= -\Omega(g,1)\Omega(1) - \Omega(g)\Omega(g,1) = (\mathrm{d}\ln Z' + (\partial \ln\rho)\circ g)R_\rho \\ &\quad + R_\rho \circ g((\mathrm{d}\ln Z')\circ g^{-1} - (\partial\ln\rho)\circ g^{-1})\circ g,\end{aligned} \tag{70}$$

where $Z = z\circ g$ and $Z' = \partial_z Z$. $R_\rho = \partial\bar{\partial}\ln\rho$ is the curvature associated to ρ. Let us define the z-component of the ghost vector field

$$\xi^z = \mathrm{d}t\,\partial_t Z\circ g^{-1}. \tag{71}$$

It is a 1-form on S^1 with values in the (conformal) vector fields of Σ. Equivalently, it is the pullback of the Maurer–Cartan form on $\mathrm{Diff}(\Sigma)$ by the loop g. One computes that

$$R_\rho(\mathrm{d}\ln Z' + (\partial\ln\rho)\circ g) = R_\rho(D_z\xi^z)\circ g, \tag{72}$$

where $D_z\xi^z = \partial_z\xi^z + \xi^z\partial_z\ln\rho$ is the covariant derivative. In the same way define

$$(\xi^{-1})^z = \mathrm{d}t\,\partial_t(z\circ g^{-1})\circ g, \tag{73}$$

one has

$$(R_\rho\circ g)((\mathrm{d}\ln Z')\circ g^{-1} - (\partial\ln\rho)\circ g^{-1})\circ g = -(R_\rho\circ g)D_z(\xi^{-1})^z. \tag{74}$$

If the loop g is the identity of Σ at $t=0$, then

$$\tilde{p}_1(g)_{t=0} = 2R_\rho D_z\xi^z \tag{75}$$

is the usual expression for the infinitesimal variation, under the ghost vector field ξ, of the vacuum functional of a field theory on Σ, i.e. a gravitational anomaly.

Now the Chern character of the signature operator Q (Section 2), contains the image of $\tilde{p}_1$ by the injection

$$\Phi : H^*(M_\Gamma) \simeq H^*(P \times_\Gamma E\Gamma) \hookrightarrow HC^*(C_c^\infty(P) \rtimes \Gamma), \tag{76}$$

where P is the bundle of all metrics (not necessarily Kähler) on the three-dimensional *real* manifold M. The topological anomaly formula then gives an *integrated* version of the infinitesimal variation (75), and is in general nonzero, provided we evaluate the anomaly on invertible *matrices* over the algebra $C_c^\infty(M) \rtimes \Gamma$.

4.3. Nontriviality of $\tilde{p}_1$

To show that $\tilde{p}_1$ and, consequently, its image in $HC^*(C_c^\infty(P) \rtimes \Gamma)$, is in general a nontrivial cohomology class, we shall construct a cycle c in the equivariant homology with compact support $H_*(M_\Gamma)$, whose evaluation on $\tilde{p}_1$ is nonzero. Since it is sufficient to do this in a particular case, let us take for Σ the Riemann sphere $\mathbb{C} \cup \{\infty\}$, and $\rho(z, \bar{z}) = 1$. Then the only nonzero component of $\tilde{p}_1$ lies in $C^2(\Gamma, \Omega^2(M))$:

$$\tilde{p}_1(g_1, g_2) = -\Omega^2(g_1 g_2, g_2, 1) = \Omega(g_1 g_2, g_2)\Omega(g_2, 1) = (\mathrm{d} \ln Z_1') \circ g_2 \, \mathrm{d} \ln Z_2'. \tag{77}$$

The equivariant homology is computed by the bicomplex $(C_{n,m})_{n,m\geq 0}$,

$$C_{n,m} = \mathbb{C}[\Gamma]^{\otimes n} \otimes \Omega_m(M), \tag{78}$$

where $\mathbb{C}[\Gamma]$ is the group ring of Γ and $\Omega_m(M)$ the space of m-dimensional de Rham currents with compact support on M. The first boundary map $\delta : C_{n,m} \to C_{n-1,m}$ is

$$\begin{aligned}\delta(g_1 \otimes \cdots \otimes g_n \otimes C) &= g_2 \otimes \cdots \otimes g_n \otimes C + \sum_{i=1}^{n} (-)^i g_1 \otimes \cdots \otimes g_i g_{i+1} \\ &\quad \otimes \cdots \otimes g_n \otimes C + (-)^{n+1} g_1 \otimes \cdots \otimes g_{n-1} \otimes g_n C,\end{aligned} \tag{79}$$

where $g_n C$ is the left action of $g_n \in \Gamma$ on the current $C \in \Omega_m(M)$ by pushforward. The second differential $\partial : C_{n,m} \to C_{n,m-1}$ is the de Rham boundary (not to be confused with the previous $\mathrm{d}z\, \partial_z$!)

$$\partial(g_1 \otimes \cdots \otimes g_n \otimes C) = (-)^n g_1 \otimes \cdots \otimes g_n \otimes \partial C. \tag{80}$$

We shall construct the cycle c as an element of $C_{1,3} \oplus C_{2,2}$. Let Γ be such that $g_1, g_2 \in \Gamma$ with

$$z \circ g_j = Z_j = \frac{\mathrm{e}^{\mathrm{i} t n_j}}{z}, \quad t \circ g_j = t, \quad j = 1, 2, \quad n_j \in \mathbb{Z}. \tag{81}$$

Choose an orientation on $M = S^1 \times \Sigma$ and let $C \in \Omega_3(M)$ be the current corresponding to the integration of 3-forms over the full cylinder

$$C = \{(z, \bar{z}, t) \in M | z\bar{z} \leq 1\}. \tag{82}$$

One checks that

$$g_i \partial C = -\partial C, \qquad g_1 g_2 C = C, \tag{83}$$

which implies that

$$c := g_1 \otimes g_2 \otimes \partial C + (g_2 - g_1 - g_1 g_2) \otimes C \tag{84}$$

represents a homology class in $H_4(M_\Gamma; \mathbb{Z})$:

$$(\partial + \delta)c = 0. \tag{85}$$

Therefore the pairing between $\tilde{p}_1$ and c is simply given by

$$\langle \tilde{p}_1, c \rangle = \int_{\partial C} \tilde{p}_1(g_1, g_2), \tag{86}$$

which gives, up to an irrelevant sign depending on the orientation, the difference $8\pi^2(n_1 - n_2)$.

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